3.4A Concavity and the 2nd Derivative Test
pp. 189–191 (1–21 odds, 49, 51, 53)

* you should be able to re-draw the [illegible] the graph. Analytical work provided but not required

1.* $\boxed{\text{concave up } (-\infty,\infty)}$
$y = x^2 - x - 2$
$y' = 2x - 1$
$\boxed{y'' = 2}$

3.* concave down
$(-2, 2)$ * work on final page
concave up
$(-\infty, -2)\ (2, \infty)$

5.* concave up
$(-\infty, -1)\ (1, \infty)$
concave down
$(-1, 1)$ * work on final page

7. $g(x) = 3x^2 - x^3$
$g'(x) = 6x - 3x^2$
$g''(x) = 6 - 6x$
$0 = 6 - 6x = 6(1-x)$
$\boxed{x = 1}$

$+$ $-$ $g''(x)$
1

$\boxed{\begin{array}{l}\text{concave up } (-\infty, 1)\\ \text{concave down } (1, \infty)\end{array}}$

9. $y = 2x - \tan x$
$y' = 2 - \sec^2 x$
$y' = -2\sec x \cdot \sec x \tan x$
$y' = -2\sec^2 x \tan x$
$-2\sec^2 x = 0 \quad \tan x = 0$
$x = 0$ $+$ $-$
0

$\boxed{\begin{array}{l}\text{concave up } (-\pi/2, 0)\\ \text{concave down } (0, \pi/2)\end{array}}$

11. $f(x) = x^3 - 6x^2 + 12x$
$f'(x) = 3x^2 - 12x + 12$
$f''(x) = 6x - 12 = 6(x-2)$
$0 = 6x - 12, \quad x = 2$

$-$ $+$
2

$\boxed{(2, 8) \text{ is a POI}}$
$\boxed{\begin{array}{l}f(x) \text{ is CCD on } (-\infty, 2)\\ \text{CCU on } (2, \infty)\end{array}}$

13. $f(x) = \frac{1}{4}x^4 - 2x^2$
$f'(x) = x^3 - 4x$
$f''(x) = 3x^2 - 4$
$0 = 3x^2 - 4$
$4/3 = x^2 \quad x = \pm\frac{2}{\sqrt{3}}$

$+$ $-$ $+$
$-2/\sqrt{3}$ $2/\sqrt{3}$

$\boxed{\begin{array}{l}\left(\frac{-2}{\sqrt{3}}, \frac{-20}{9}\right)\left(\frac{-2}{\sqrt{3}}, \frac{-20}{9}\right)\\ f(x) \text{ is CCD on } \left(-\sqrt{\frac{4}{3}}, \sqrt{\frac{4}{3}}\right)\\ \text{CCU on } \left(-\infty, -\sqrt{\frac{4}{3}}\right)\left(\sqrt{\frac{4}{3}}, \infty\right)\end{array}}$

15. $f(x) = x(x-4)^3$
$f'(x) = x \cdot 3(x-4)^2 + (x-4)^3$
$f'(x) = (x-4)^2(3x + x - 4)$
$f'(x) = (x-4)^2(4x-4) =$
$f'(x) = 4(x-1)(x-4)^2$
$f''(x) = 4[(x-1)(2)(x-4) + (x-4)^2]$
$f''(x) = 4[2(x-1)(x-4) + (x-4)^2]$
$f''(x) = 4(x-4)[2x - 2 + x - 4]$
$f''(x) = 4(x-4)(3x-6)$
$f''(x) = 4(x-4)(3)(x-2)$
$f''(x) = 12(x-4)(x-2)$
$0 = 12(x-4)(x-2)$
$x = 2, 4$

$+$ $-$ $+$
2 4

$\boxed{\begin{array}{l}\text{POI } (2, 16)(4, 0)\\ f(x) \text{ is CCD on } (2, 4)\\ f(x) \text{ is CCU on } (-\infty, 2) \text{ and } (4, \infty)\end{array}}$

17. $f(x) = x\sqrt{x+3}$

$f'(x) = x \cdot \frac{1}{2}(x+3)^{-1/2} + (x+3)^{1/2}$

$f'(x) = \frac{x}{2\sqrt{x+3}} + \sqrt{x+3}$

$f'(x) = \frac{x + 2(x+3)}{2\sqrt{x+3}} = \frac{3x+6}{2\sqrt{x+3}}$

$f'(x) = \frac{3(x+2)}{2\sqrt{x+3}}$

$f''(x) = \frac{3}{2}\left[\frac{(x+3)^{1/2}(1) - (x+2)\cdot\frac{1}{2}(x+3)^{-1/2}}{x+3}\right]$

$f''(x) = \frac{3}{2}\left[\frac{(x+3)^{1/2} - \frac{(x+2)}{2(x+3)^{1/2}}}{(x+3)}\right]$

$f''(x) = \frac{3}{2}\left[\frac{\frac{2(x+3)-(x+2)}{2(x+3)^{1/2}}}{(x+3)}\right]$

$f''(x) = \frac{3}{2}\left[\frac{2x+6-x-2}{2(x+3)^{3/2}}\right] = \frac{3}{2}\left[\frac{x+4}{2(x+3)^{3/2}}\right]$

$f''(x) = \frac{3(x+4)}{4(x+3)^{3/2}}$

Domain of $f(x)$ is $[-3, \infty)$
$f''(x) > 0$ on the entire domain except for $x = -3$ because $f''(-3)$ is undefined
no points of inflection
$f(x)$ is concave up on $(-3, \infty)$

19. $f(x) = \frac{x}{x^2+1}$ $f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2}$

$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

$f''(x) = \frac{(x^2+1)^2(-2x) - (1-x^2)\cdot 2(x^2+1)(2x)}{(x^2+1)^4}$

$f''(x) = \frac{-2x(x^2+1)^2 - 4x(x^2+1)(1-x^2)}{(x^2+1)^4}$

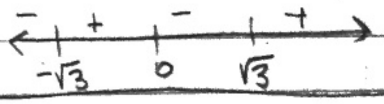
$f''(x) = \frac{(x^2+1)\left[-2x(x^2+1) - 4x(1-x^2)\right]}{(x^2+1)^4}$

$f''(x) = \frac{(x^2+1)\left[-2x^3-2x-4x+4x^3\right]}{(x^2+1)^4}$

$f''(x) = \frac{(x^2+1)(2x^3-6x)}{(x^2+1)^4}$

$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$

$f''(x) = 0$ when $x = 0, \sqrt{3}, -\sqrt{3}$



$f(x)$ is CCD on $(-\infty, -\sqrt{3}) + (0, \sqrt{3})$
CCU on $(-\sqrt{3}, 0) + (\sqrt{3}, \infty)$
$f(x)$ has points of inflection at
$(-\sqrt{3}, \frac{-\sqrt{3}}{4})(0,0)(\sqrt{3}, \frac{\sqrt{3}}{4})$

21. $f(x) = \sin\left(\frac{x}{2}\right)$ $[0, 4\pi]$

$f'(x) = \cos\left(\frac{x}{2}\right)\frac{1}{2}$

$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$

$f''(x) = \frac{1}{2} \cdot -\sin\left(\frac{x}{2}\right) \cdot \frac{1}{2}$

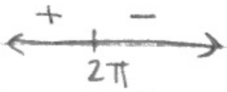
$f''(x) = \frac{-1}{4}\sin\left(\frac{x}{2}\right)$

$0 = \frac{-1}{4}\sin\left(\frac{x}{2}\right)$

$\frac{x}{2} = 0, \ \frac{x}{2} = \pi, \ \frac{x}{2} = 3\pi, \ \frac{x}{2} = 4\pi$

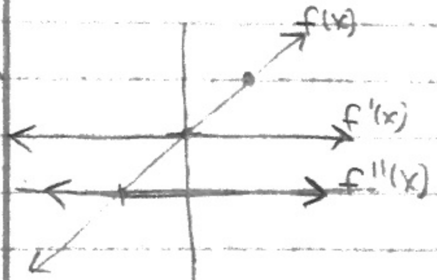
$x = 0 \quad x = 2\pi \quad x = 6\pi$

$\boxed{\text{POI @ } x = 2\pi}$

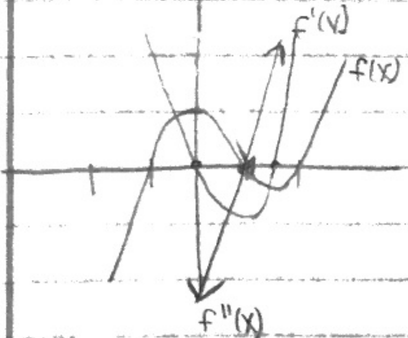


$\boxed{\begin{array}{l} f(x) \text{ is CCU on } (0, 2\pi) \\ \text{CCD on } (2\pi, 4\pi) \end{array}}$

49.



51.



53. $f(2) = f(4) = 0$

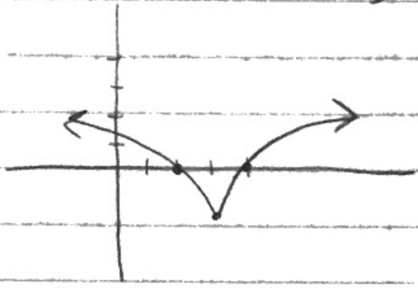
$f(3)$ is defined

$f'(x) < 0$ if $x < 3$ decreasing

$f'(3)$ DNE sharp point

$f'(x) > 0$ if $x > 3$ increasing

$f''(x) < 0$ if $x \neq 3$ CCD



3. $f(x) = 24(x^2+12)^{-1}$

$f'(x) = -24(x^2+12)^{-2}(2x)$

$f'(x) = \frac{-48x}{(x^2+12)^2}$

$f''(x) = \frac{(x^2+12)^2(-48) - (-48x)(2)(x^2+12)(2x)}{(x^2+12)^4}$

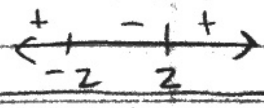
$f''(x) = \frac{-48(x^2+12)^2 + 192x^2(x^2+12)}{(x^2+12)^4}$

$f''(x) = \frac{(x^2+12)[-48(x^2+12) + 192x^2]}{(x^2+12)^4}$

$f''(x) = \frac{-48x^2 - 576 + 192x^2}{(x^2+12)^3} = \frac{144x^2 - 576}{(x^2+12)^3}$

$f''(x) = \frac{144(x^2-4)}{(x^2+12)^3}$

$f''(x) = 0$ when $x = \pm 2$



$\boxed{\begin{array}{l} f(x) \text{ is concave up on } (-\infty, -2) + (2, \infty) \\ f(x) \text{ is concave down on } (-2, 2) \end{array}}$

5. $f(x) = \frac{x^2+1}{x^2-1}$

$f'(x) = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$

$f'(x) = \frac{2x^3-2x-2x^3-2x}{(x^2-1)^2}$

$f'(x) = \frac{-4x}{(x^2-1)^2}$

$f''(x) = \frac{(x^2-1)^2(-4) - (-4x)\cdot 2(x^2-1)(2x)}{(x^2-1)^4}$

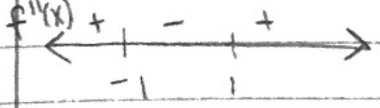
$f''(x) = \frac{(x^2-1)\left[-4(x^2-1)+16x^2\right]}{(x^2-1)^4}$

$f''(x) = \frac{-4x^2+4+16x^2}{(x^2-1)^3}$

$f''(x) = \frac{12x^2+4}{(x^2-1)^3}$

$f''(x) = \frac{4(3x^2+1)}{(x^2-1)^3}$ $f''(x) = 0$ when $x = \pm 1$

there are no points of inflection
because $f''(x)$ DNE at points
not in the domain of f



f(x) is concave up on $(-\infty, -1)$ + $(1, \infty)$
concave down on $(-1, 1)$